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Abstract

This article proposes a framework to handle multiattribute group decision making problems with incomplete pairwise comparison preference over decision alternatives where qualitative and quantitative attribute values are furnished as linguistic variables and crisp numbers, respectively. Attribute assessments are then converted to interval-valued intuitionistic fuzzy numbers (IVIFNs) to characterize fuzziness and uncertainty in the evaluation process. Group consistency and inconsistency indices are introduced for incomplete pairwise comparison preference relations on alternatives provided by the decision-makers (DMs). By minimizing the group inconsistency index under certain constraints, an auxiliary linear programming model is developed to obtain unified attribute weights and an interval-valued intuitionistic fuzzy positive ideal solution (IVIFPIS). Attribute weights are subsequently employed to calculate distances between alternatives and the IVIFPIS for ranking alternatives. An illustrative example is provided to demonstrate the applicability and effectiveness of this method.

Keywords: Multi-attribute group decision making (MAGDM), interval-valued intuitionistic fuzzy numbers (IVIFNs), linear programming, group consistency and inconsistency

1. Introduction

When facing a decision situation, a decision-maker (DM) often has to evaluate a

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finite set of alternatives against multiple attributes. This process can be conveniently modeled as a multiattribute decision making (MADM) problem. Several formal procedures have been proposed to deal with MADM such as the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) (Hwang & Yoon, 1981) and the Linear Programming Technique for Multidimensional Analysis of Preference (LINMAP) (Srinivasan & Shocker, 1973). The LINMAP proves to be a practical and useful technique for determining attribute weights and a positive-ideal solution based on a DM's pairwise comparisons of alternatives. In the traditional LINMAP, performance ratings are known precisely and given as crisp values. Under many practical decision situations, it is hard, if not impossible, to obtain exact assessment values due to inherent vagueness and uncertainty in human judgment. As such, Zadeh (1965) puts forward a powerful paradigm, fuzzy set theory, to handle ambiguity information that often arises in human decision processes. The LINMAP has subsequently been extended to handle MADM with fuzzy judgment data (Li & Yang, 2004).

In Zadeh's fuzzy set, an element's membership to a particular set is defined as a real value μ between 0 and 1 and its nonmembership is implied to be $1-\mu$. This extension of traditional binary logic provides a powerful framework to characterize vagueness and uncertainty. The treatment of nonmembership as a complement of membership essentially omits a DM's hesitation in the decision making process. To facilitate further characterization of uncertainty and vagueness, Atanassov (1986) introduces intuitionistic fuzzy sets (IFSs), depicted by real-valued membership, nonmembership, and hesitancy functions. Due to its capability of accommodating hesitation in human decision processes, IFSs have been widely recognized as flexible and practical tools for tackling imprecise and uncertain decision information (Xu & Cai, 2010) and have been widely applied to the field of decision modeling. For instance, Li (2005) proposes a linear programming method to handle MADM using IFSs; Wei (2010) develops an intuitionistic fuzzy weighted geometric operator-based approach to solve multi-attribute group decision making (MAGDM) problems; Li et al. (2010) extend the LINMAP method to solve MAGDM with intuitionistic fuzzy information.

An IFS is characterized by real-valued membership and nonmembership functions defined on [0, 1], and the hesitancy function can be easily derived based on the aforesaid two functions. In some decision situations with highly uncertain and imprecise judgment, it could pose a significant challenge to require that membership and nonmembership be identified as exact values. To address this issue, IFSs are further extended to interval-valued intuitionistic fuzzy sets (IVIFSs) (Atanassov and Gargov, 1989) where membership and nonmembership are represented as interval-valued functions. Since its inception, significant research has been conducted to develop and enrich the IVIFS theory, such as investigations on the correlation and correlation coefficients of IVIFSs (Bustince & Burillo 1995; Hong, 1998; Hung & Wu, 2002), fuzzy cross entropy of IVIFSs (Ye, 2011), relationships between IFSs, *L*-fuzzy sets, interval-valued fuzzy sets and IVIFSs (Deschrijver, 2007; Deschrijver, 2008; Deschrijver & Kerre, 2007), similarity measures of IVIFSs (Wei, Wang, & Zhang, 2011; Xu & Chen, 2008), and comparison of the interval-valued intuitionistic numbers (IVIFNs) (Li & Wang, 2010; Wang, Li, & Wang, 2009; Xu, 2007). Thanks to their advantage in coping with uncertain decision data, IVIFSs have been widely applied to decision models with multiple attributes (Li, 2010a, b; Wang, Li & Wang, 2009; Park et al., 2011; Li, 2011; Wang, Li, & Xu, 2011; Wei, 2010, 2011; Xu, 2007; Xu & Yager, 2007, 2008; Xu et al., 2011). Recently, researchers started to address MAGDM problems involving IVIFS decision data. For instance, Park et al. (2009) investigate group decision problems based on correlation coefficients of IVIFSs. Xu (2010) introduces certain IVIFN relations and operations and proposes a distance-based method for group decisions. Ye (2010) develops a MAGDM method with IVIFNs to solve the partner selection problem of a virtual enterprise under incomplete information. Yue (2011) puts forward an approach to aggregate interval numbers into IVIFNs for group decisions. Chen et al. (2011) propose a framework to tackle MAGDM problem based on interval-valued intuitionistic fuzzy preference relations and interval-valued intuitionistic fuzzy decision matrices.

To the authors' knowledge, little research has been carried out to handle MAGDM problems in which attribute values are converted to IVIFNs with unknown attribute weights and incomplete pairwise comparison preference relations on alternatives. In this research, the focus is to further extend the LINMAP method and develop a new approach to MAGDM problems with IVIFN decision data. In this paradigm, it is assumed that raw decision data are furnished as linguistic variables (for qualitative attributes) and numerical values (for quantitative attributes), then IVIFNs are constructed to reflect fuzziness and uncertainty contained in attribute assessment values and DMs' subjective judgment. Group consistency and inconsistency indices are defined for pairwise comparison preference relations on alternatives. A linear program is proposed for deriving the interval-valued intuitionistic fuzzy positive ideal solution (IVIFPIS) and attribute weights. The distances of alternatives to the IVIFPIS are calculated to determine their ranking orders for individual DMs. Finally, a group ranking order can be generated using the Borda function (Hwang & Yoon, 1981). An earlier version of this paper was presented at a conference and published in the proceedings [Wang, Wang & Li, 2011]. This manuscript has significantly expanded the research reported therein by refining the modeling process, addressing certain technical deficiency, and furnishing two theorems to reveal useful properties of the proposed framework.

The remainder of the paper is organized as follows. Section 2 provides preliminaries on IVIFSs and Euclidean distance between IVIFNs. Section 3 formulates the MAGDM problem with IVIFNs and defines group consistency and inconsistency indices. Section 4 proposes an approach to handle MAGDM problems with IVIFNs, and a linear program is established to estimate the IVIFPIS and attribute weights. Section 5 presents a numerical example to demonstrate how to apply the proposed approach, followed by some concluding remarks in Section 6.

2. Preliminaries

Let Z be a fixed nonempty universe set, an IFS A in Z is an object in the following form (Atanassov, 1986):

$$A = \{< z, \mu_A(z), \nu_A(z) > | \, z \in Z\},$$

where $\mu_A : Z \to [0,1]$ and $\nu_A : Z \to [0,1]$, satisfying $0 \le \mu_A(z) + \nu_A(z) \le 1$, $\forall z \in Z$.

$\mu_A(z)$ and $\nu_A(z)$ denote, respectively, the degree of membership and nonmembership of element z to set A. In addition, for each IFS A in Z, $\pi_A(z) = 1 - \mu_A(z) - \nu_A(z)$ is often referred to as its intuitionistic fuzzy index, representing the degree of indeterminacy of z to A. Obviously, $0 \le \pi_A(z) \le 1$ for every $z \in Z$.

Given that the degrees of membership and nonmembership are sometimes difficult to be derived with exact values, Atanassov and Gargov (1989) extend IFSs to interval-valued intuitionistic fuzzy sets (IVIFSs) that allow membership and nonmembership functions to assume interval values.

Let $D([0,1])$ be the set of all closed subintervals of the unit interval [0, 1], an IVIFS $\tilde{A}$ over Z is defined as

$$\tilde{A} = \{< z, \tilde{\mu}_{\tilde{A}}(z), \tilde{v}_{\tilde{A}}(z) > | z \in Z\},$$

where $\tilde{\mu}_{\tilde{A}} : Z \to D([0,1])$, $\tilde{v}_{\tilde{A}} : Z \to D([0,1])$, and $0 \le \sup(\tilde{\mu}_{\tilde{A}}(z)) + \sup(\tilde{v}_{\tilde{A}}(z)) \le 1$ for any $z \in Z$.

The intervals $\tilde{\mu}_{\tilde{A}}(z)$ and $\tilde{v}_{\tilde{A}}(z)$ define, respectively, the degree of membership and nonmembership of z to A. Thus for each $z \in Z$, the difference from an IFS is that $\tilde{\mu}_{\tilde{A}}(z)$ and $\tilde{v}_{\tilde{A}}(z)$ are closed intervals, and their lower and upper bounds are denoted by $\tilde{\mu}_{\tilde{A}}^L(z), \tilde{\mu}_{\tilde{A}}^U(z), \tilde{v}_{\tilde{A}}^L(z)$ and $\tilde{v}_{\tilde{A}}^U(z)$, respectively. Therefore, the IVIFS $\tilde{A}$ can be equivalently expressed as

$$\tilde{A} = \{< z, [\tilde{\mu}_{\tilde{A}}^L(z), \tilde{\mu}_{\tilde{A}}^U(z)], [\tilde{v}_{\tilde{A}}^L(z), \tilde{v}_{\tilde{A}}^U(z)] > | z \in Z\},$$

where $\tilde{\mu}_{\tilde{A}}^U(z) + \tilde{v}_{\tilde{A}}^U(z) \le 1, 0 \le \tilde{\mu}_{\tilde{A}}^L(z) \le \tilde{\mu}_{\tilde{A}}^U(z) \le 1, 0 \le \tilde{v}_{\tilde{A}}^L(z) \le \tilde{v}_{\tilde{A}}^U(z) \le 1$.

Similar to IFSs, an interval intuitionistic fuzzy index of an element $z \in Z$ is expressed as

$$\tilde{\pi}_{\tilde{A}}(z) = [\tilde{\pi}_{\tilde{A}}^L(z), \tilde{\pi}_{\tilde{A}}^U(z)] = [1 - \tilde{\mu}_{\tilde{A}}^U(z) - \tilde{v}_{\tilde{A}}^U(z), 1 - \tilde{\mu}_{\tilde{A}}^L(z) - \tilde{v}_{\tilde{A}}^L(z)],$$

which gives the range of hesitancy degree of element z to set $\tilde{A}$.

If each of the intervals $\tilde{\mu}_{\tilde{A}}(z)$ and $\tilde{v}_{\tilde{A}}(z)$ contains only a single value, i.e., for every $z \in Z$, $\tilde{\mu}_{\tilde{A}}^L(z) = \tilde{\mu}_{\tilde{A}}^U(z)$ and $\tilde{v}_{\tilde{A}}^L(z) = \tilde{v}_{\tilde{A}}^U(z)$, then the given IVIFS $\tilde{A}$ is reduced to a regular IFS.

For an IVIFS $\tilde{A}$ and a given z, the pair $(\tilde{\mu}_{\tilde{A}}(z), \tilde{v}_{\tilde{A}}(z))$ is called an interval-valued intuitionistic fuzzy number (IVIFN) (Wang, Li, & Wang, 2009; Wang, Li, & Xu, 2011; Xu, 2007; Xu & Yager, 2008). For convenience, we denote an IVIFN by $([a,b],[c,d])$, where $[a,b] \in D([0,1])$, $[c,d] \in D([0,1])$ and $b + d \le 1$.

Xu and Yager (2009) introduce the normalized Hamming distance considering interval intuitionistic fuzzy index between IVIFSs. Here, a normalized Euclidean distance is introduced to facilitate the discussion in Section 3.

Let $\tilde{\alpha}_1 = ([a_1, b_1],[c_1, d_1])$ and $\tilde{\alpha}_2 = ([a_2, b_2],[c_2, d_2])$ be any two IVIFNs, then a normalized Euclidean distance between $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ can be defined as:

$$d(\tilde{\alpha}_1, \tilde{\alpha}_2) = (\frac{1}{4}((a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2 + (d_1 - d_2)^2 + (\pi^l_{\tilde{\alpha}_1} - \pi^l_{\tilde{\alpha}_2})^2 + (\pi^u_{\tilde{\alpha}_1} - \pi^u_{\tilde{\alpha}_2})^2))^{1/2} \tag{2.1}$$

where $\pi^l_{\tilde{\alpha}_1} = 1 - b_1 - d_1, \pi^u_{\tilde{\alpha}_1} = 1 - a_1 - c_1, \pi^l_{\tilde{\alpha}_2} = 1 - b_2 - d_2, \pi^u_{\tilde{\alpha}_2} = 1 - a_2 - c_2$.

3. An MAGDM problem and group consistency measurement

This section presents an MAGDM problem with IVIFNs and defines group consistency and inconsistency indices.

3.1 An MAGDM framework with IVIFN decision data

Given n feasible decision alternatives x_i (i =1, 2, …, n) and m qualitative or quantitative attributes a_j (j = 1, 2, …, m). Denote the alternative set by $X = \{x_1, x_2, \ldots, x_n\}$ and the attribute set by $A = \{a_1, a_2, \ldots, a_m\}$. The attribute set A can be divided into two mutually exclusive and collectively exhaustive subsets: A_1 and A_2, representing the subset of qualitative and quantitative attributes, respectively. It is natural that $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \varnothing$, where $\varnothing$ is the empty set. Depending on the decision purpose, an MAGDM problem could be defined as finding the best alternative(s) from all feasible choices or obtaining a ranking for all alternatives based on the information provided by a group of DMs $D = \{d_1, d_2, \ldots, d_q\}$.

Assume that DM $d_p \in D$ assesses each alternative $x_i \in X$ on each qualitative attribute $a_j \in A_1$ as a linguistic variable. These linguistic assessments are then converted into IVIFNs, $\tilde{r}^p_{ij} = ([a^{1p}_{ij}, b^{1p}_{ij}],[c^{1p}_{ij}, d^{1p}_{ij}])$ (i = 1, 2, …, n, p = 1, 2, …, q). The intervals $[a^{1p}_{ij}, b^{1p}_{ij}]$ and $[c^{1p}_{ij}, d^{1p}_{ij}]$ are the degree of satisfaction (or membership) and the degree of non-satisfaction (or nonmembership) of x_i on the qualitative attribute a_j with respect to a fuzzy concept "excellence", and satisfy the following conditions: $[a^{1p}_{ij}, b^{1p}_{ij}] \in D([0,1])$,

Table 1. A conversion table between linguistic variables and IVIFNs

Linguistic terms	IVIFNs
Very Good (VG)	([0.90,0.95],[0.02,0.05])
Good (G)	([0.70,0.75],[0.20,0.25])
Fair (F)	([0.50,0.55],[0.40,0.45])
Poor (P)	([0.20,0.25],[0.70,0.75])
Very Poor (VP)	([0.02,0.05],[0.90,0.95])

$[c_{ij}^{1p}, d_{ij}^{1p}] \in D([0,1])$ and $b_{ij}^{1p} + d_{ij}^{1p} \le 1$. Table 1 furnishes a conversion table between linguistic variables and their corresponding IVIFNs used in the case study in Section 5.

For each quantitative attribute $a_j \in A_2$, it is assumed that each alternative $x_i \in X$ is assessed as a numerical value, denoted by f_{ij}^p. Generally speaking, numerical assessments on different attributes often assume different units (e.g., kilograms for weight and kilometers for distance). In addition, for the same numerical value f_{ij}^p, different DMs may have different degrees of satisfaction (or membership) and non-satisfaction (or nonmembership) assessment. As such, it is desirable to convert a numerical value f_{ij}^p to dimensionless relative degrees of satisfaction and non-satisfaction, reflecting both objective measurement and DM d_p's subjective assessment.

Quantitative attributes are often classified into two types: benefit and cost attributes. Denote the benefit attribute set by A_2^b and the cost attribute set by A_2^c. One way to define the relative degree of satisfaction interval $[a_{ij}^{2p}, b_{ij}^{2p}]$ for a numerical value f_{ij}^p is given as follows:

$$\begin{cases} a_{ij}^{2p} = \beta_j^{pl}(f_{ij}^p - f_{jp}^{\min})/(f_{jp}^{\max} - f_{jp}^{\min}) \\ b_{ij}^{2p} = \beta_j^{pu}(f_{ij}^p - f_{jp}^{\min})/(f_{jp}^{\max} - f_{jp}^{\min}) \end{cases} \quad if\ a_j \in A_2^b,$$
$$\begin{cases} a_{ij}^{2p} = \beta_j^{pl}(f_{jp}^{\max} - f_{ij}^p)/(f_{jp}^{\max} - f_{jp}^{\min}) \\ b_{ij}^{2p} = \beta_j^{pu}(f_{jp}^{\max} - f_{ij}^p)/(f_{jp}^{\max} - f_{jp}^{\min}) \end{cases} \quad if\ a_j \in A_2^c, \tag{3.1}$$

where $f_{jp}^{\max} = \max\{f_{ij}^p \mid i = 1, 2, \ldots, n\}$, $f_{jp}^{\min} = \min\{f_{ij}^p \mid i = 1, 2, \ldots, n\}$ and the parameter $\bar{\beta}_j^p = [\beta_j^{pl}, \beta_j^{pu}] \in D([0,1])$ is given by DM d_p (p = 1, 2, …, q) according to its expected goals and needs in the decision situation, reflecting the DM's relative degree of satisfaction (or membership) for the best assessment on attribute $a_j \in A_2$ (maximum for a benefit attribute or minimum for a cost attribute).

It is obvious that $[a_{ij}^{2p}, b_{ij}^{2p}] \in D([0,1])$ and the larger the relative degree interval $[a_{ij}^{2p}, b_{ij}^{2p}]$, the more satisfying alternative x_i is with respect to attribute a_j.

For a numerical value f_{ij}^{p} (i =1, 2, …, n, $a_j \in A_2$), let

$$f_{ij}^{'p} = \kappa_j^p f_{ij}^p + \lambda_j^p, \tag{3.2}$$

where $\kappa_j^p > 0$ and λ_j^p are constants given by the DM d_p (p =1, 2, …, q). The purpose of introducing this linear transformation formula is to accommodate the case that DM d_p may adopt a different rating system for a quantitative attribute $a_j \in A_2$. Next, Theorem 3.1 establishes that the relative degree of satisfaction interval for a numerical value f_{ij}^{p} remains the same for its converted value $f_{ij}^{'p}$ under the transformation relation (3.2).

Theorem 3.1 For a numerical assessment f_{ij}^{p} and its converted value $f_{ij}^{'p}$ based on Eq. (3.2), denote their relative degree of satisfaction intervals by $[a_{ij}^{2p}, b_{ij}^{2p}]$ and $[a'^{2p}_{ij}, b'^{2p}_{ij}]$, then $a_{ij}^{2p} = a'^{2p}_{ij}$ and $b_{ij}^{2p} = b'^{2p}_{ij}$.

Proof. Since

$$\begin{aligned} f_{jp}^{'\max} &= \max\{\kappa_j^p f_{ij}^p + \lambda_j^p \mid i = 1, 2, \ldots, n\} \\ &= \kappa_j^p \max\{f_{ij}^p \mid i = 1, 2, \ldots, n\} + \lambda_j^p \\ &= \kappa_j^p f_{jp}^{\max} + \lambda_j^p \end{aligned}$$

and

$$\begin{aligned} f_{jp}^{'\min} &= \min\{\kappa_j^p f_{ij}^p + \lambda_j^p \mid i = 1, 2, \ldots, n\} \\ &= \kappa_j^p \min\{f_{ij}^p \mid i = 1, 2, \ldots, n\} + \lambda_j^p \\ &= \kappa_j^p f_{jp}^{\min} + \lambda_j^p \end{aligned}$$

Then,

$$
\begin{cases}
a'^{2p}_{ij} = \beta^{pl}_j (f'^{p}_{ij} - f'^{\min}_{jp}) / (f'^{\max}_{jp} - f'^{\min}_{jp}) \\
\quad = \beta^{pl}_j (\kappa^p_j f^p_{ij} + \lambda^p_j - (\kappa^p_j f^{\min}_{jp} + \lambda^p_j)) / ((\kappa^p_j f^{\max}_{jp} + \lambda^p_j) - (\kappa^p_j f^{\min}_{jp} + \lambda^p_j)) \\
\quad = \beta^{pl}_j (f^p_{ij} - f^{\min}_{jp}) / (f^{\max}_{jp} - f^{\min}_{jp}) = a^{2p}_{ij} \\
b'^{2p}_{ij} = \beta^{pu}_j (f'^{p}_{ij} - f^{\min}_{jp}) / (f'^{\max}_{jp} - f'^{\min}_{jp}) \\
\quad = \beta^{pu}_j (\kappa^p_j f^p_{ij} + \lambda^p_j - (\kappa^p_j f^{\min}_{jp} + \lambda^p_j)) / ((\kappa^p_j f^{\max}_{jp} + \lambda^p_j) - (\kappa^p_j f^{\min}_{jp} + \lambda^p_j)) \\
\quad = \beta^{pu}_j (f^p_{ij} - f^{\min}_{jp}) / (f^{\max}_{jp} - f^{\min}_{jp}) = b^{2p}_{ij}
\end{cases}
\qquad if\ a_j \in A^b_2,
$$

$$
\begin{cases}
a'^{2p}_{ij} = \beta^{pl}_j (f'^{\max}_{jp} - f'^{p}_{ij}) / (f'^{\max}_{jp} - f'^{\min}_{jp}) \\
\quad = \beta^{pl}_j (\kappa^p_j f^{\max}_{jp} + \lambda^p_j - (\kappa^p_j f^p_{ij} + \lambda^p_j)) / ((\kappa^p_j f^{\max}_{jp} + \lambda^p_j) - (\kappa^p_j f^{\min}_{jp} + \lambda^p_j)) \\
\quad = \beta^{pl}_j (f^{\max}_{jp} - f^p_{ij}) / (f^{\max}_{jp} - f^{\min}_{jp}) = a^{2p}_{ij} \\
b'^{2p}_{ij} = \beta^{pu}_j (f'^{\max}_{jp} - f'^{p}_{ij}) / (f'^{\max}_{jp} - f'^{\min}_{jp}) \\
\quad = \beta^{pu}_j (\kappa^p_j f^{\max}_{jp} + \lambda^p_j - (\kappa^p_j f^p_{ij} + \lambda^p_j)) / ((\kappa^p_j f^{\max}_{jp} + \lambda^p_j) - (\kappa^p_j f^{\min}_{jp} + \lambda^p_j)) \\
\quad = \beta^{pu}_j (f^{\max}_{jp} - f^p_{ij}) / (f^{\max}_{jp} - f^{\min}_{jp}) = b^{2p}_{ij}
\end{cases}
\qquad if\ a_j \in A^c_2,
$$

The proof of Theorem 3.1 is thus completed. ■

Theorem 3.1 guarantees that Eq. (3.1) always yields the same relative degree of satisfaction interval for a numerical assessment even if it is converted to a different rating system as long as the conversion process follows the linear relationship in Eq. (3.2).

Similarly, assume that DM d_p (p =1, 2, …, q) gives its relative degree of non-satisfaction interval as $[\hat{c}^{2p}_j, \hat{d}^{2p}_j]$ for the best assessment on attribute $a_j \in A_2$ (maximum value $f^{\max}_{jp}$ for a benefit attribute or minimum value $f^{\min}_{jp}$ for a cost attribute), where $\hat{d}^{2p}_j + \beta^{pu}_j \le 1$ for all $a_j \in A_2$.

Let

$$
\gamma^{pl}_j = \begin{cases} \dfrac{\tilde{c}^{2p}_j}{1-\beta^{pu}_j} & \beta^{pu}_j < 1 \\ 0 & \beta^{pu}_j = 1 \end{cases}
\qquad
\gamma^{pu}_j = \begin{cases} \dfrac{\tilde{d}^{2p}_j}{1-\beta^{pu}_j} & \beta^{pu}_j < 1 \\ 0 & \beta^{pu}_j = 1 \end{cases}
\tag{3.3}
$$

Obviously, $\gamma_j^{pl} \le \gamma_j^{pu}$ and $[\gamma_j^{pl}, \gamma_j^{pu}] \in D([0,1])$. Denote $\overline{\gamma}_j^p \triangleq [\gamma_j^{pl}, \gamma_j^{pu}]$, then DM d_p's relative degree of non-satisfaction interval $[c_{ij}^{2p}, d_{ij}^{2p}]$ for the numerical value f_{ij}^p can be computed by the following formula:

$$[c_{ij}^{2p}, d_{ij}^{2p}] = (1-b_{ij}^{2p})\overline{\gamma}_j^p = [\gamma_j^{pl}(1-b_{ij}^{2p}), \gamma_j^{pu}(1-b_{ij}^{2p})] \tag{3.4}$$

As $0 \le \gamma_j^{pu} \le 1$ and $0 \le b_{ij}^{2p} \le 1$, it follows that $0 \le b_{ij}^{2p} + \gamma_j^{pu}(1-b_{ij}^{2p}) \le b_{ij}^{2p} + 1 - b_{ij}^{2p} = 1$, we have $0 \le b_{ij}^{2p} + d_{ij}^{2p} \le 1$. Therefore, Eqs. (3.1) and (3.4) ensure that a numerical assessment f_{ij}^p is transformed into an IVIFN, $([a_{ij}^{2p}, b_{ij}^{2p}],[c_{ij}^{2p}, d_{ij}^{2p}])$.

Let

$$\tilde{r}_{ij}^p = ([a_{ij}^p, b_{ij}^p],[c_{ij}^p, d_{ij}^p]) = \begin{cases} ([a_{ij}^{1p}, b_{ij}^{1p}],[c_{ij}^{1p}, d_{ij}^{1p}]) & \text{if } a_j \in A_1 \\ ([a_{ij}^{2p}, b_{ij}^{2p}],[c_{ij}^{2p}, d_{ij}^{2p}]) & \text{if } a_j \in A_2 \end{cases} \tag{3.5}$$

where i = 1,2, …, n and j = 1, 2, …, m. Thus, an MAGDM problem with IVIFNs can be concisely expressed in an IVIFN matrix format as follows:

$$\tilde{R}^p = (\tilde{r}_{ij}^p)_{n\times m} = \left(([a_{ij}^p, b_{ij}^p],[c_{ij}^p, d_{ij}^p])\right)_{n\times m}, \quad (p = 1, 2, \dots, q) \tag{3.6}$$

3.2 Group consistency and inconsistency

In an MAGDM problem, different attribute weights reflect their varying importance in selecting the final alternative. Let $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$ be the unknown attribute weight vector, where $\omega_j \ge 0$, $j = 1,2,\dots,m$, and the weights are often normalized to one, i.e. $\sum_{j=1}^m \omega_j = 1$. Denote the unknown interval-valued intuitionistic fuzzy positive ideal solution (IVIFPIS) by $x^* = (\tilde{r}_1^*, \tilde{r}_2^*, \dots, \tilde{r}_m^*)^T$, where $\tilde{r}_j^* = ([a_j^*, b_j^*],[c_j^*, d_j^*])$ $(j = 1,2,\dots,m)$ are IVIFNs. Then the weighted average of squared Euclidean distance between DM d_p's assessment vector $x_i^p = (\tilde{r}_{i1}^p, \tilde{r}_{i2}^p, \dots, \tilde{r}_{im}^p)$ and the IVIFPIS $x^* = (\tilde{r}_1^*, \tilde{r}_2^*, \dots, \tilde{r}_m^*)^T$ can be defined as follows:

$$S_i^p = \sum_{j=1}^m \omega_j [d(\tilde{r}_{ij}^p, \tilde{r}_j^*)]^2 \tag{3.7}$$

By (2.1), S_i^p can be expanded as:

$$S_i^p = \frac{1}{4}\sum_{j=1}^{m}\omega_j[(a_{ij}^p - a_j^*)^2 + (b_{ij}^p - b_j^*)^2 + (c_{ij}^p - c_j^*)^2 + (d_{ij}^p - d_j^*)^2 + (\pi_{ij}^{pl} - \pi_j^{*l})^2 + (\pi_{ij}^{pu} - \pi_j^{*u})^2] \tag{3.8}$$

where $\pi_{ij}^{pl} = 1 - b_{ij}^p - d_{ij}^p$, $\pi_{ij}^{pu} = 1 - a_{ij}^p - c_{ij}^p$, $\pi_j^{*l} = 1 - b_j^* - d_j^*$ and $\pi_j^{*u} = 1 - a_j^* - c_j^*$.

Let

$$\begin{aligned} F_{ij}^p &= \frac{1}{4}[(a_{ij}^p)^2 + (b_{ij}^p)^2 + (c_{ij}^p)^2 + (d_{ij}^p)^2 + (\pi_{ij}^{pl})^2 + (\pi_{ij}^{pu})^2 - 2\pi_{ij}^{pl} - 2\pi_{ij}^{pu}], \\ C_{ij}^p &= \frac{1}{2}(-a_{ij}^p + \pi_{ij}^{pu}),\ G_{ij}^p = \frac{1}{2}(-b_{ij}^p + \pi_{ij}^{pl}), \\ H_{ij}^p &= \frac{1}{2}(-c_{ij}^p + \pi_{ij}^{pu}),\ T_{ij}^p = \frac{1}{2}(-d_{ij}^p + \pi_{ij}^{pl}) \end{aligned} \tag{3.9}$$

and

$$\hat{a}_j = \omega_j a_j^*, \hat{b}_j = \omega_j b_j^*,\ \hat{c}_j = \omega_j c_j^*,\ \hat{d}_j = \omega_j d_j^* \tag{3.10}$$

for each i =1, 2, …, n, j =1, 2, …, m. Then S_i^p can be written as:

$$S_i^p = \sum_{j=1}^{m}\omega_j F_{ij}^p + \sum_{j=1}^{m}\hat{a}_j C_{ij}^p + \sum_{j=1}^{m}\hat{b}_j G_{ij}^p + \sum_{j=1}^{m}\hat{c}_j H_{ij}^p + \sum_{j=1}^{m}\hat{d}_j T_{ij}^p + \frac{1}{4}\sum_{j=1}^{m}\omega_j[(a_j^*)^2 + (b_j^*)^2 + (c_j^*)^2 + (d_j^*)^2 + (\pi_j^{*l})^2 + (\pi_j^{*u})^2] \tag{3.11}$$

If the weight vector ω and the IVIFPIS x^* are given by the DMs, then S_i^p (i = 1, 2, …, n) can be calculated by using (3.11). A ranking of alternatives can thus be conveniently obtained for DM d_p based on S_i^p. However, in this paper, it is conceived that the weight vector ω and the IVIFPIS x^* are not provided by the DMs. Instead, based on incomplete pairwise comparisons of alternatives, a model is proposed to generate a best compromise alternative as the solution that has the shortest distance to the IVIFPIS. To accomplish this goal, consistency and inconsistency indices are introduced based on S_i^p and incomplete pairwise preference relations on alternatives furnished by the DMs.

Assume that DM $d_p \in D$ (p =1, 2, …, q) provides its comparison preference relations on alternatives as $\Omega^p = \{(k,t) \mid x_k \succsim_p x_t, k,t \in \{1,2,\ldots,n\}\}$, where $x_k \succsim_p x_t$ indicates that DM d_p prefers x_k to x_t or is indifferent between x_k and x_t.

By (3.7), $S_t^p \geq S_k^p$ means that alternative x_k is closer to the IVIFPIS x^* compared to alternative x_t. In this case, the ranking order of alternatives x_k and x_t implied by the normalized Euclidean distance is $x_k \succsim_p x_t$. If DM d_p furnishes the same pairwise comparison result for these two alternatives, i.e., $(k,t) \in \Omega^p$, the ranking is called consistent. Otherwise, if the computed distance reveals $S_t^p < S_k^p$, but the ranking order furnished by the DM is $x_k \succsim_p x_t$, this ranking is referred to as inconsistent. This inconsistency indicates that the weights and IVIFPIS x^* are not chosen properly. Next, the consistency index of DM d_p is introduced as follows:

$$E^p = \sum_{(k,t)\in\Omega^p} \max\{0, S_t^p - S_k^p\} \tag{3.12}$$

and the group consistency index is thus calculated as:

$$E = \sum_{p=1}^{q} E^p = \sum_{p=1}^{q} \sum_{(k,t)\in\Omega^p} \max\{0, S_t^p - S_k^p\} \tag{3.13}$$

Similarly, the inconsistency index of DM d_p is defined as:

$$B^p = \sum_{(k,t)\in\Omega^p} \max\{0, S_k^p - S_t^p\} \tag{3.14}$$

and the group inconsistency index is determined as:

$$B = \sum_{p=1}^{q} B^p = \sum_{p=1}^{q} \sum_{(k,t)\in\Omega^p} \max\{0, S_k^p - S_t^p\} \tag{3.15}$$

Let

$$F_{ijs}^p = F_{ij}^p - F_{sj}^p, C_{ijs}^p = C_{ij}^p - C_{sj}^p, G_{ijs}^p = G_{ij}^p - G_{sj}^p, H_{ijs}^p = H_{ij}^p - H_{sj}^p, T_{ijs}^p = T_{ij}^p - T_{sj}^p \tag{3.16}$$

for each i, s =1, 2, …, n, j =1, 2, …, m. Then it follows from (3.11) that

$$\begin{aligned} &\max\{0, S_i^p - S_s^p\} - \max\{0, S_s^p - S_i^p\} = S_i^p - S_s^p \\ &= \sum_{j=1}^{m} \omega_j F_{ijs}^p + \sum_{j=1}^{m} \hat{a}_j C_{ijs}^p + \sum_{j=1}^{m} \hat{b}_j G_{ijs}^p + \sum_{j=1}^{m} \hat{c}_j H_{ijs}^p + \sum_{j=1}^{m} \hat{d}_j T_{ijs}^p \end{aligned} \tag{3.17}$$

for each i, s =1, 2, …, n. From (3.13), (3.15) and (3.17), one can obtain that

$$\begin{aligned}E-B&=\sum_{p=1}^{q}\sum_{(k,t)\in\Omega^p}(\max\{0,S_t^p-S_k^p\}-\max\{0,S_k^p-S_t^p\})\\&=\sum_{p=1}^{q}\sum_{(k,t)\in\Omega^p}(S_t^p-S_k^p)\\&=\sum_{j=1}^{m}\omega_j\left(\sum_{p=1}^{q}\sum_{(k,t)\in\Omega^p}F_{tjk}^p\right)+\sum_{j=1}^{m}\hat{a}_j\left(\sum_{p=1}^{q}\sum_{(k,t)\in\Omega^p}C_{tjk}^p\right)+\sum_{j=1}^{m}\hat{b}_j\left(\sum_{p=1}^{q}\sum_{(k,t)\in\Omega^p}G_{tjk}^p\right)+\\&\quad\sum_{j=1}^{m}\hat{c}_j\left(\sum_{p=1}^{q}\sum_{(k,t)\in\Omega^p}H_{tjk}^p\right)+\sum_{j=1}^{m}\hat{d}_j\left(\sum_{p=1}^{q}\sum_{(k,t)\in\Omega^p}T_{tjk}^p\right).\end{aligned}\tag{3.18}$$

Denote

$$F_j=\sum_{p=1}^{q}\sum_{(k,t)\in\Omega^p}F_{tjk}^p,C_j=\sum_{p=1}^{q}\sum_{(k,t)\in\Omega^p}C_{tjk}^p,G_j=\sum_{p=1}^{q}\sum_{(k,t)\in\Omega^p}G_{tjk}^p,H_j=\sum_{p=1}^{q}\sum_{(k,t)\in\Omega^p}H_{tjk}^p,T_j=\sum_{p=1}^{q}\sum_{(k,t)\in\Omega^p}T_{tjk}^p\tag{3.19}$$

Then, Eq. (3.18) can be simply rewritten as follows:

$$E-B=\sum_{j=1}^{m}\omega_jF_j+\sum_{j=1}^{m}\hat{a}_jC_j+\sum_{j=1}^{m}\hat{b}_jG_j+\sum_{j=1}^{m}\hat{c}_jH_j+\sum_{j=1}^{m}\hat{d}_jT_j\tag{3.20}$$

4 A linear programming approach to the MAGDM problem

As the group inconsistency index B reflects the overall inconsistency between the derived Euclidean distance and the DMs' judgment, the smaller the B, the better the model characterizes the DMs' decision rationales. Therefore, a sensible attribute weight vector $\omega=(\omega_1,\omega_2,\ldots,\omega_m)^T$ and IVIFPIS x^* is to minimize the group inconsistency index B (Li et al. (2010) apply the similar treatment to handle multiattribute group decision making with intuitionistic fuzzy sets). Based on this consideration, the following optimization model is established to determine ω and x^*:

$$\begin{aligned}&\min\{B\}\\&s.t.\ E-B\ge h\\&\quad b_j^*+d_j^*\le 1, a_j^*\le b_j^*, c_j^*\le d_j^*\quad (j=1,2,\ldots,m)\\&\quad a_j^*\ge 0, b_j^*\ge 0, c_j^*\ge 0, d_j^*\ge 0\quad (j=1,2,\ldots,m)\\&\quad \omega_j\ge 0\quad (j=1,2,\ldots,m).\end{aligned}\tag{4.1}$$

where h is a positive number that is expected to reflect by how much the consistency index should exceed the inconsistency index for the group of DMs.

Utilizing (3.15) and (3.20), model (4.1) can be converted to the following mathematical programming model:

$$\begin{aligned}
&\min\{\sum_{p=1}^{q}\sum_{(k,t)\in\Omega^p}\max\{0,S_k^p-S_t^p\}\}\\
&s.t\sum_{j=1}^{m}\omega_j F_j+\sum_{j=1}^{m}\hat{a}_j C_j+\sum_{j=1}^{m}\hat{b}_j G_j+\sum_{j=1}^{m}\hat{c}_j H_j+\sum_{j=1}^{m}\hat{d}_j T_j\ \ge h\\
&\quad b_j^*+d_j^*\le 1,\ a_j^*\le b_j^*,c_j^*\le d_j^*\quad (j=1,2,\ldots,m)\\
&\quad a_j^*\ge 0,b_j^*\ge 0,c_j^*\ge 0,d_j^*\ge 0\quad (j=1,2,\ldots,m)\\
&\quad \omega_j\ge 0\quad (j=1,2,\ldots,m).
\end{aligned}\tag{4.2}$$

For each pair of alternatives $(k,t)\in\Omega^p$, let $\xi_{kt}^p\triangleq\max\left(0,S_k^p-S_t^p\right)$, then $\xi_{kt}^p\ge-(S_t^p-S_k^p)$, i.e., $(S_t^p-S_k^p)+\xi_{kt}^p\ge 0$. It follows from (3.17) that

$$\sum_{j=1}^{m}\omega_j F_{tjk}^p+\sum_{j=1}^{m}\hat{a}_j C_{tjk}^p+\sum_{j=1}^{m}\hat{b}_j G_{tjk}^p+\sum_{j=1}^{m}\hat{c}_j H_{tjk}^p+\sum_{j=1}^{m}\hat{d}_j T_{tjk}^p+\xi_{kt}^p\ge 0\tag{4.3}$$

As $\hat{a}_j=\omega_j a_j^*,\hat{b}_j=\omega_j b_j^*,\ \hat{c}_j=\omega_j c_j^*,\ \hat{d}_j=\omega_j d_j^*$ (j = 1, 2, …, m), one can confirm that $\hat{a}_j\le\hat{b}_j,\ \hat{c}_j\le\hat{d}_j$ since $a_j^*\le b_j^*,c_j^*\le d_j^*$, and $\hat{b}_j+\hat{d}_j\le\omega_j$ due to $b_j^*+d_j^*\le 1$ for j = 1, 2, …, m. By incorporating (4.3) as a constraint, the nonlinear model (4.2) is transformed to the following linear program by treating ξ_{kt}^p as free decision variables:

$$\begin{aligned}
&\min\{\sum_{p=1}^{q}\sum_{(k,t)\in\Omega^p}\xi_{kt}^p\}\\
&s.t.\sum_{j=1}^{m}\omega_j F_j+\sum_{j=1}^{m}\hat{a}_j C_j+\sum_{j=1}^{m}\hat{b}_j G_j+\sum_{j=1}^{m}\hat{c}_j H_j+\sum_{j=1}^{m}\hat{d}_j T_j\ \ge h\\
&\quad\sum_{j=1}^{m}\omega_j F_{tjk}^p+\sum_{j=1}^{m}\hat{a}_j C_{tjk}^p+\sum_{j=1}^{m}\hat{b}_j G_{tjk}^p+\sum_{j=1}^{m}\hat{c}_j H_{tjk}^p+\sum_{j=1}^{m}\hat{d}_j T_{tjk}^p+\xi_{kt}^p\ge 0\\
&\quad((k,t)\in\Omega^p;\ p=1,2,\ldots,q)\\
&\quad\xi_{kt}^p\ge 0\quad((k,t)\in\Omega^p;\ p=1,2,\ldots,q)\\
&\quad\hat{b}_j+\hat{d}_j\le\omega_j,\ \hat{a}_j\le\hat{b}_j,\hat{c}_j\le\hat{d}_j\quad (j=1,2,\ldots,m)\\
&\quad\hat{a}_j\ge 0,\hat{b}_j\ge 0,\hat{c}_j\ge 0,\hat{d}_j\ge 0\quad (j=1,2,\ldots,m)\\
&\quad\omega_j\ge 0\quad (j=1,2,\ldots,m).
\end{aligned}\tag{4.4}$$

It is apparent that the optimal solution of (4.4) depends on the parameter h. Denote the optimal solution by $(\omega_1^0(h),\omega_2^0(h),...,\omega_m^0(h))$, $(\hat{a}_1^0(h),\hat{a}_2^0(h),...,\hat{a}_m^0(h))$, $(\hat{b}_1^0(h),\hat{b}_2^0(h),...,\hat{b}_m^0(h))$, $(\hat{c}_1^0(h),\hat{c}_2^0(h),...,\hat{c}_m^0(h))$, $(\hat{d}_1^0(h),\hat{d}_2^0(h),...,\hat{d}_m^0(h))$, and

$((\xi_{kt}^{0p}(h))_{(k,t)\in\Omega^p})$ $(p = 1, 2, \ldots, q)$, respectively.

Given the constraints $\hat{b}_j + \hat{d}_j \le \omega_j, \hat{a}_j \le \hat{b}_j, \hat{c}_j \le \hat{d}_j, \hat{a}_j \ge 0, \hat{b}_j \ge 0, \hat{c}_j \ge 0, \hat{d}_j \ge 0$ $(j = 1, 2, \ldots, m)$ in (4.4), it follows that $\hat{a}_j = 0, \hat{b}_j = 0, \hat{c}_j = 0, \hat{d}_j = 0$ if $\omega_j = 0$, and $\frac{\hat{b}_j}{\omega_j} + \frac{\hat{d}_j}{\omega_j} \le 1$ if $\omega_j > 0$. Therefore, the optimal values of $a_j^*, b_j^* c_j^*, d_j^*$ (j = 1, 2, …, m), denoted by $a_j^{*0}(h), b_j^{*0}(h), c_j^{*0}(h), d_j^{*0}(h)$, can be computed using (3.10) as follows:

$$a_j^{*0}(h) = \begin{cases} \dfrac{\hat{a}_j^0(h)}{\omega_j^0(h)} & \text{if } \omega_j^0(h) > 0 \\ 0 & \text{if } \omega_j^0(h) = 0 \end{cases}, b_j^{*0}(h) = \begin{cases} \dfrac{\hat{b}_j^0(h)}{\omega_j^0(h)} & \text{if } \omega_j^0(h) > 0 \\ 0 & \text{if } \omega_j^0(h) = 0 \end{cases},$$
$$c_j^{*0}(h) = \begin{cases} \dfrac{\hat{c}_j^0(h)}{\omega_j^0(h)} & \text{if } \omega_j^0(h) > 0 \\ 0 & \text{if } \omega_j^0(h) = 0 \end{cases}, d_j^{*0}(h) = \begin{cases} \dfrac{\hat{d}_j^0(h)}{\omega_j^0(h)} & \text{if } \omega_j^0(h) > 0 \\ 0 & \text{if } \omega_j^0(h) = 0 \end{cases} \quad (4.5)$$

It is clear that $\omega_j^0(h) = 0$ corresponds to the case that attribute a_j does not contribute to the distance S_i^p between alternative x_i and the IVIFPIS. In this case, a_j is irrelevant in determining DM d_p's preference.

It is easy to verify that $[a_j^{*0}(h), b_j^{*0}(h)] \in D([0,1]), [c_j^{*0}(h), d_j^{*0}(h)] \in D([0,1])$ and $b_j^{*0}(h) + d_j^{*0}(h) \le 1$. Let $\tilde{r}_j^{*0}(h) = ([a_j^{*0}(h), b_j^{*0}(h)], [c_j^{*0}(h), d_j^{*0}(h)])$ (j = 1 ,2,…, m). Thus, an optimal IVIFPIS, denoted by $x^{*0}(h) = (\tilde{r}_1^{*0}(h), \tilde{r}_2^{*0}(h), \ldots, \tilde{r}_m^{*0}(h))^T$, is determined.

As linear program (4.4) does not include a weight normalization condition, the optimal weight vector $(\omega_1^0(h), \omega_2^0(h), ..., \omega_m^0(h))^T$ should then be normalized as

$$(\omega_1^0(h) / \sum_{j=1}^{m} \omega_j^0(h), \omega_2^0(h) / \sum_{j=1}^{m} \omega_j^0(h), ..., \omega_m^0(h) / \sum_{j=1}^{m} \omega_j^0(h))^T \quad (4.6)$$

Once the optimal weights and the IVIFPIS are obtained from (4.5) and (4.6), the distance between each alternative and the IVIFPIS can be calculated for each DM d_p as S_i^p based on (3.8), from which a ranking of all alternatives can be derived accordingly for DM d_p (p = 1, 2, …, q).

Linear program (4.4) possesses a fine property that makes it convenient to apply the proposed method.

Theorem 4.1 If h in the first constraint of the linear program (4.4) is changed to a different positive number, the optimal IVIFPIS determined by (4.5) and the normalized weight vector calculated by (4.6) remain optimal.

Proof. Let $\hat{h}>0$ and $\hat{h}\neq h$. Multiplying the objective function and both sides of the constraints in (4.4) by $\frac{\hat{h}}{h}$ yields the following linear program:

$$
\begin{aligned}
&\min\{\sum_{p=1}^{q}\sum_{(k,t)\in\Omega^p}\xi_{kt}^{p}\frac{\hat{h}}{h}\}\\
&s.t.\sum_{j=1}^{m}\omega_j\frac{\hat{h}}{h}F_j+\sum_{j=1}^{m}\hat{a}_j\frac{\hat{h}}{h}C_j+\sum_{j=1}^{m}\hat{b}_j\frac{\hat{h}}{h}G_j+\sum_{j=1}^{m}\hat{c}_j\frac{\hat{h}}{h}H_j+\sum_{j=1}^{m}\hat{d}_j\frac{\hat{h}}{h}T_j\ \ge h\frac{\hat{h}}{h}=\hat{h}\\
&\quad\sum_{j=1}^{m}\omega_j\frac{\hat{h}}{h}F_{tjk}^{p}+\sum_{j=1}^{m}\hat{a}_j\frac{\hat{h}}{h}C_{tjk}^{p}+\sum_{j=1}^{m}\hat{b}_j\frac{\hat{h}}{h}G_{tjk}^{p}+\sum_{j=1}^{m}\hat{c}_j\frac{\hat{h}}{h}H_{tjk}^{p}+\sum_{j=1}^{m}\hat{d}_j\frac{\hat{h}}{h}T_{tjk}^{p}+\xi_{kt}^{p}\frac{\hat{h}}{h}\ge 0\\
&\quad((k,t)\in\Omega^p;\ p=1,2,\ldots,q)\\
&\quad\xi_{kt}^{p}\frac{\hat{h}}{h}\ge 0\quad((k,t)\in\Omega^p;\ p=1,2,\ldots,q)\\
&\quad\hat{b}_j\frac{\hat{h}}{h}+\hat{d}_j\frac{\hat{h}}{h}\le\omega_j\frac{\hat{h}}{h},\hat{a}_j\frac{\hat{h}}{h}\le\hat{b}_j\frac{\hat{h}}{h},\hat{c}_j\frac{\hat{h}}{h}\le\hat{d}_j\frac{\hat{h}}{h}\quad(j=1,2,\ldots,m)\\
&\quad\hat{a}_j\frac{\hat{h}}{h}\ge 0,\hat{b}_j\frac{\hat{h}}{h}\ge 0,\hat{c}_j\frac{\hat{h}}{h}\ge 0,\hat{d}_j\frac{\hat{h}}{h}\ge 0\quad(j=1,2,\ldots,m)\\
&\quad\omega_j\frac{\hat{h}}{h}\ge 0\quad(j=1,2,\ldots,m).
\end{aligned}
$$

Let $\xi_{kt}^{'p}\triangleq\xi_{kt}^{p}\frac{\hat{h}}{h},\omega_j^{'}\triangleq\omega_j\frac{\hat{h}}{h},\hat{a}_j^{'}\triangleq\hat{a}_j\frac{\hat{h}}{h},\hat{b}_j^{'}\triangleq\hat{b}_j\frac{\hat{h}}{h},\hat{c}_j^{'}\triangleq\hat{c}_j\frac{\hat{h}}{h},$ and $\hat{d}_j^{'}\triangleq\hat{d}_j\frac{\hat{h}}{h},$ it is apparent that the aforesaid linear program is identical to (4.4) except for the relabeled decision variables and the right-hand value of the first constraint. Then $\omega_j^{'0}(\hat{h})=\frac{\hat{h}}{h}\omega_j^{0}(h)$, $\hat{a}_j^{'0}(\hat{h})=\frac{\hat{h}}{h}\hat{a}_j^{0}(h)$, $\hat{b}_j^{'0}(\hat{h})=\frac{\hat{h}}{h}\hat{b}_j^{0}(h)$, $\hat{c}_j^{'0}(\hat{h})=\frac{\hat{h}}{h}\hat{c}_j^{0}(h)$, and $\hat{d}_j^{'0}(\hat{h})=\frac{\hat{h}}{h}\hat{d}_j^{0}(h)$ ($j=1,2,\ldots,m$).

Therefore, we have

$$
\tilde{r}_j^{'*0}(\hat{h})=([\frac{\hat{a}_j^{0}(\hat{h})}{\omega_j^{0}(\hat{h})},\frac{\hat{b}_j^{0}(\hat{h})}{\omega_j^{0}(\hat{h})}],[\frac{\hat{c}_j^{0}(\hat{h})}{\omega_j^{0}(\hat{h})},\frac{\hat{d}_j^{0}(\hat{h})}{\omega_j^{0}(\hat{h})}])=([\frac{\hat{a}_j^{0}(h)}{\omega_j^{0}(h)},\frac{\hat{b}_j^{0}(h)}{\omega_j^{0}(h)}],[\frac{\hat{c}_j^{0}(h)}{\omega_j^{0}(h)},\frac{\hat{d}_j^{0}(h)}{\omega_j^{0}(h)}])=\tilde{r}_j^{*0}(h)
$$

and $\omega_j^{'0}(\hat{h}) / \sum_{j=1}^{m} \omega_j^{'0}(\hat{h}) = (\frac{\hat{h}}{h}\omega_j^0(h)) / \sum_{j=1}^{m} \frac{\hat{h}}{h}\omega_j^0(h) = \omega_j^0(h) / \sum_{j=1}^{m} \omega_j^0(h)$ (j= 1, 2, …, m). ■

Theorem 4.1 indicates that the parameter value h in the linear program (4.4) is irrelevant in determining the optimal IVIFPIS and normalized weight vector. The implication is that an analyst can select any positive h value to calibrate the model.

Based on the aforesaid analyses, we are now in a position to formulate an interval-valued intuitionistic fuzzy approach to MAGDM as described in the following steps.

Step 1. Convert linguistic assessments on alternative $x_i \in X$ to appropriate IVIFNs for qualitative attributes $a_j \in A_1$.

Step 2. Calculate corresponding IVIFNs for numerical assessments on alternative $x_i \in X$ for quantitative attributes $a_j \in A_2$ as per (3.1) and (3.4).

Step 3. Construct the IVIFN decision matrix $\tilde{R}^p = (\tilde{r}_{ij}^p)_{n \times m} = \left(([a_{ij}^p, b_{ij}^p],[c_{ij}^p, d_{ij}^p])\right)_{n \times m}$ for DM d_p (p=1, 2, …, q).

Step 4. Establish the linear programming model (4.4) based on the incomplete pairwise comparison preference relations furnished by the DMs.

Step 5. Obtain the optimal values $\omega_j^0(h)$, $\hat{a}_j^0(h)$, $\hat{b}_j^0(h)$, $\hat{c}_j^0(h)$ and $\hat{d}_j^0(h)$ (j=1, 2, …, m) by solving (4.4) with any given parameter $h > 0$.

Step 6. Calculate the optimal normalized weight vector as per (4.6).

Step 7. Determine the optimal IVIFPIS $x^{*0}(h) = (\tilde{r}_1^{*0}(h), \tilde{r}_2^{*0}(h), \ldots, \tilde{r}_m^{*0}(h))^T$ as per (4.5).

Step 8. Compute the weighted average of squared Euclidean distances S_i^p between alternatives x_i and the IVIFPIS $x^{*0}(h)$ as per (3.8) (i = 1, 2, …, n, p = 1, 2, …, q).

Step 9. Rank all alternatives for DM d_p (p = 1, 2, …, q) according to an increasing order of their distances S_i^p (i =1, 2, …, n).

Step 10. Rank all alternatives for the group using the Borda function (Hwang & Yoon, 1981) and the best alternative is the one with the smallest Borda scores.

5 An illustrative example

This section presents an MAGDM problem about recommending undergraduate students for graduate admission to demonstrate how to apply the proposed approach.

Without loss of generality, assume that there are three committee members (i.e., DMs) d_1, d_2, and d_3, and four students x_1, x_2, x_3, and x_4 as the finalists after preliminary screening. All DMs agree to evaluate these candidates against four attributes, academic records (a_1), college English test Band 6 score (a_2), teamwork skills (a_3), and research potentials (a_4). a_1 is assessed based the cumulative grade point average (GPA), and a_2 is assessed out of 710 points with a minimum qualifying level of 425 points. a_1 and a_2 are both benefit quantitative attributes. a_3 and a_4 can be well characterized as qualitative attributes and their ratings can be easily expressed as linguistic variables. This example assumes that the group has agreed to assess qualitative attributes on five linguistic terms as given in Table 1, which also provides a conversion table between linguistic terms and IVIFNs. Assume that the three committee members have furnished their assessments of the four candidates on the four attributes as shown in Table 2.

Table 2. Raw decision data furnished by the DMs

Experts	Students	Attributes			
		a_1	a_2	a_3	a_4
d_1	x_1	88	550	F	VG
	x_2	96	520	P	F
	x_3	92	580	G	G
	x_4	90	500	F	F
d_2	x_1	88	550	G	G
	x_2	96	520	P	F
	x_3	92	580	F	VG
	x_4	90	500	F	F
d_3	x_1	88	550	F	VG
	x_2	96	520	P	F
	x_3	92	580	F	F
	x_4	90	500	G	F

Assume further that the DMs provide their incomplete pariwise comparison preference relations on the four candidates as follows:

$$\Omega^1 = \{(1,2),(3,1),(2,4),(4,3)\},\ \Omega^2 = \{(2,1),(4,3),(1,3)\},\ \Omega^3 = \{(3,1),(2,3),(4,1)\}.$$

From Table 2, one can easily verify that $f_{1p}^{\max} = 96$, $f_{1p}^{\min} = 88$, $f_{2p}^{\max} = 580$, $f_{2p}^{\min} = 500$ (p = 1, 2, 3). For this particular example, the assessment values on the two quantitative attributes are common for the three DMs given that they are simply taken from the four candidates' historical records. However, it is worth noting that the proposed model in this paper is able to handle the case where each DM provides different assessments for quantitative attributes.

For the same quantitative assessment, it is understandable that different DMs may have different opinions on how well it satisfies a particular attribute. For instance, what percentage grade can be converted to a letter grade of A? The answer to this question depends on what grade conversion scale is adopted by an instructor. Therefore, it is sensible that each DM may have different degrees of satisfaction and non-satisfaction for the same quantitative assessment. It is assumed that DM d_p, p = 1, 2, 3, provide their degrees of satisfaction for $f_{1p}^{\max} = 96$ as $\bar{\beta}_1^1 = \left[\beta_1^{1l}, \beta_1^{1u}\right] = [0.90, 0.95]$, $\bar{\beta}_1^2 = \left[\beta_1^{2l}, \beta_1^{2u}\right] =$ $[0.85, 0.90]$, and $\bar{\beta}_1^3 = \left[\beta_1^{3l}, \beta_1^{3u}\right] = [0.86, 0.92]$; degrees of non-satisfaction as $\left[\hat{c}_1^{21}, \hat{d}_1^{21}\right]$ $= \left[0.02, 0.03\right]$, $\left[\hat{c}_1^{22}, \hat{d}_1^{22}\right] = \left[0.05, 0.08\right]$, and $\left[\hat{c}_1^{23}, \hat{d}_1^{23}\right] = \left[0.05, 0.07\right]$, respectively. Similarly, assume that DM d_p, p = 1, 2, 3, furnish their degree of satisfaction for $f_{2p}^{\max} = 580$ as $\bar{\beta}_2^1 = \left[\beta_2^{1l}, \beta_2^{1u}\right] = [0.88, 0.92]$, $\bar{\beta}_2^2 = \left[\beta_2^{2l}, \beta_2^{2u}\right] = [0.9, 0.92]$, and $\bar{\beta}_2^3 =$ $\left[\beta_2^{3l}, \beta_2^{3u}\right] =$ $[0.85, 0.90]$, and $\left[\hat{c}_2^{21}, \hat{d}_2^{21}\right] = \left[0.03, 0.06\right]$, $\left[\hat{c}_2^{22}, \hat{d}_2^{22}\right] =$ $\left[0.03, 0.05\right]$, and $\left[\hat{c}_2^{23}, \hat{d}_2^{23}\right] =$ $\left[0.05, 0.07\right]$, respectively.

Based on (3.1), one can derive each DM's degrees of satisfaction for the four candidates against the two quantitative attributes as the first intervals in every cell of the first two columns in Tables 3, 4, and 5.

By using (3.3), one can determine: $\bar{\gamma}_1^1 = [0.40, 0.60]$, $\bar{\gamma}_1^2 = [0.50, 0.80]$, $\bar{\gamma}_1^3 =$ $[0.625, 0.875]$, $\bar{\gamma}_2^1 = [0.375, 0.75]$, $\bar{\gamma}_2^2 = [0.375, 0.625]$, $\bar{\gamma}_2^3 = [0.50, 0.70]$. According to (3.4), each DM's degrees of nonsatisfaction for all candidates for the two quantitative attributes are derived as the second intervals in every cell of the first two columns in Tables 3, 4, and 5.

As per Table 1, the linguistic assessments on the two qualitative attributes can be converted to interval-valued intuitionistic fuzzy data. The result is shown in the last two columns of the decision matrices for DM d_p (p = 1 ,2 ,3) in Tables 3, 4, and 5:

Table 3. Interval-valued intuitionistic fuzzy decision matrix for DM d_1 $\tilde{R}^1$

	a_1	a_2	a_3	a_4
x_1	([0.0000,0.0000],[0.4000,0.6000])	([0.5500,0.5750],[0.1594,0.3188])	([0.50,0.55],[0.40,0.45])	([0.90,0.95],[0.02,0.05])
x_2	([0.9000,0.9500],[0.0200,0.0300])	([0.2200,0.2300],[0.2888,0.5775])	([0.20,0.25],[0.70,0.75])	([0.50,0.55],[0.40,0.45])
x_3	([0.4500,0.4750],[0.2100,0.3150])	([0.8800,0.9200],[0.0300,0.0600])	([0.70,0.75],[0.20,0.25])	([0.70,0.75],[0.20,0.25])
x_4	([0.2250,0.2375],[0.3050,0.4575])	([0.0000,0.0000],[0.3750,0.7500])	([0.50,0.55],[0.40,0.45])	([0.50,0.55],[0.40,0.45])

Table 4. Interval-valued intuitionistic fuzzy decision matrix for DM d_2 $\tilde{R}^2$

	a_1	a_2	a_3	a_4
x_1	([0.0000,0.0000],[0.5000,0.8000])	([0.5625,0.5750],[0.1594,0.2656])	([0.70,0.75],[0.20,0.25])	([0.70,0.75],[0.20,0.25]))
x_2	([0.8500,0.9000],[0.0500,0.0800])	([0.2250,0.2300],[0.2888,0.4813])	([0.20,0.25],[0.70,0.75])	([0.50,0.55],[0.40,0.45])
x_3	([0.4250,0.4500],[0.2750,0.4400])	([0.9000,0.9200],[0.0300,0.0500])	([0.50,0.55],[0.40,0.45])	([0.90,0.95],[0.02,0.05])
x_4	([0.2125,0.2250],[0.3875,0.6200])	([0.0000,0.0000],[0.3750,0.6250])	([0.50,0.55],[0.40,0.45])	([0.50,0.55],[0.40,0.45])

Table 5. Interval-valued intuitionistic fuzzy decision matrix for DM d_3 $\tilde{R}^3$

	a_1	a_2	a_3	a_4
x_1	([0.0000,0.0000],[0.6250,0.8750])	([0.5313,0.5625],[0.2188,0.3063])	([0.50,0.55],[0.40,0.45])	([0.90,0.95],[0.02,0.05])
x_2	([0.8600,0.9200],[0.0500,0.0700])	([0.2125,0.2250],[0.3875,0.5425])	([0.20,0.25],[0.70,0.75])	([0.50,0.55],[0.40,0.45])
x_3	([0.4300,0.4600],[0.3375,0.4725])	([0.8500,0.9000],[0.0500,0.0700])	([0.50,0.55],[0.40,0.45])	([0.50,0.55],[0.40,0.45])
x_4	([0.2150,0.2300],[0.4813,0.6783])	([0.0000,0.0000],[0.5000,0.7000])	([0.70,0.75],[0.20,0.25])	([0.50,0.55],[0.40,0.45])

It can be seen from the interval-valued intuitionistic fuzzy decision matrix $\tilde{R}^1$ that DM d_1's degrees of satisfaction and non-satisfaction for x_2 on a_1 are computed as $[0.9000, 0.9500]$ and $[0.0200, 0.0300]$ rather than $[1,1]$ and $[0,0]$ although x_2 reaches the maximum $f_{11}^{\max} = 96$. This conversion process presumably reflects that DM d_1 is not completely satisfied with candidate a_1's cumulative GPA $f_{11}^{\max} = 96$ although this student achieves the highest GPA among the four candidates. Similarly, $\tilde{r}_{31}^1$ indicates that DM d_1's degrees of satisfaction and non-satisfaction for x_3 on a_1 are [0.45, 0.475] and [0.21, 0.315], respectively. This converted IVIFN assessment points to a hesitancy degree of [0.21, 0.34] for DM d_1.

As per Theorem 4.1, the parameter h in (4.4) can be arbitrarily selected without affecting the optimal normalized weights and IVIFPIS. By setting $h=1$, solving model (4.4) yields the following optimal solution:

$$(\omega_1^0,\omega_2^0,\omega_3^0,\omega_4^0)^T=(701.5739,1030.2918,394.9273,485.3135)^T,$$

$$(\hat{a}_1^0,\hat{a}_2^0,\hat{a}_3^0,\hat{a}_4^0)^T=(290.1888,343.5678,129.3340,166.3520)^T,$$

$$(\hat{b}_1^0,\hat{b}_2^0,\hat{b}_3^0,\hat{b}_4^0)^T=(393.3232,494.7810,208.7332,267.7018)^T,$$

$$(\hat{c}_1^0,\hat{c}_2^0,\hat{c}_3^0,\hat{c}_4^0)^T=(45.5403,167.0847,47.5738,47.6016)^T,$$

$$(\hat{d}_1^0,\hat{d}_2^0,\hat{d}_3^0,\hat{d}_4^0)^T=(104.4404,230.7817,110.3714,120.8024)^T.$$

By using (4.6), one can obtain the optimal normalized weight vector as $(0.2686,0.3944,0.1512,0.1858)^T$.

As per (4.5), the optimal IVIFPIS is determined as

$$\begin{aligned}x^{*0}=(&([0.4136,0.5606],[0.0649,0.1489]),([0.3335,0.4802],[0.1622,0.2240]),\\&([0.3275,0.5284],[0.1205,0.2795]),([0.3428,0.5516],[0.0981,0.2489]))^T\end{aligned}.$$

According to (3.8), the weighted average of squared Euclidean distances S_i^p (i =1, 2, …, 4, p = 1, 2, 3) between x_i and the IVIFPIS can be calculated as follows:

$$\begin{aligned}&S_1^1=0.120194,\ S_2^1=0.120192,\ S_3^1=0.120181,\ S_4^1=0.120159,\\&S_1^2=0.123826,\ S_2^2=0.105802,\ S_3^2=0.146691,\ S_4^2=0.123683,\\&S_1^3=0.157639,\ S_2^3=0.117237,\ S_3^3=0.125221,\ S_4^3=0.148978.\end{aligned}$$

Since $S_1^1 > S_2^1 > S_3^1 > S_4^1, S_3^2 > S_1^2 > S_4^2 > S_2^2, S_1^3 > S_4^3 > S_3^3 > S_2^3$, then the ranking orders of the four alternatives for the three DMs are derived as $x_4 \succ_1 x_3 \succ_1 x_2 \succ_1 x_1$, $x_2 \succ_2 x_4 \succ_2 x_1 \succ_2 x_3$ and $x_2 \succ_3 x_3 \succ_3 x_4 \succ_3 x_1$, respectively, where $x_k \succ_p x_t$ indicates that DM d_p prefers x_k to x_t or ranks x_k higher than x_t.

Using the Borda function (Hwang & Yoon, 1981), Borda scores of the four candidates can be determined as shown in the last column of Table 6.

The final group ranking of the four alternatives can thus be obtained as $x_2 \succ x_4 \succ x_3 \succ x_1$.

Table 6. Borda scores of the four candidates

Candidate	Decision-maker			Borda score
	d_1	d_2	d_3	
x_1	3	2	3	8
x_2	2	0	0	2
x_3	1	3	1	5
x_4	0	1	2	3

6 CONCLUSIONS

In a typical MAGDM problem, both quantitative and qualitative attributes are often involved and assessed with imprecise data and subjective judgment. This article first proposes mechanisms for converting numerical quantitative assessments and linguistic qualitative values into IVIFN decision data. Based on incomplete pairwise comparison preference relations furnished by the DMs, group consistency and inconsistency indices are introduced. The converted IVIFN decision data and group consistency and inconsistency indices are then employed to establish a linear programming model for determining unified attribute weights and IVIFPIS. An illustrative numerical example is developed to demonstrate how to apply the proposed framework.

Current research assumes that qualitative and quantitative attributes are assessed as linguistic terms and numerical values, respectively. Additional research is needed to handle the case when the corresponding assessments are expressed as interval linguistic variables and interval numbers. Moreover, the current linear program (4.4) assumes that each DM has the same influence over the decision process. It is a worthy topic to address the situation that different DMs exert distinct weights on choosing the final alternative.

REFERENCES

Atanassov, K. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems, 20*, 87-96.

Atanassov, K. & Gargov, G. (1989). Interval-valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems 31*, 343-349.

Bustince, H. & Burillo, P. (1995). Correlation of interval-valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems, 74*, 237-244.

Chen, T.Y., Wang, H.P., & Lu, Y.Y. (2011). A multicriteria group decision-making approach based on interval-valued intuitionistic fuzzy sets: A comparative perspective. *Expert Systems with Applications*, 38, 7647–7658.

Deschrijver, G. (2007). Arithmetic operators in interval-valued fuzzy set theory. *Information Sciences, 177*, 2906-2924.

Deschrijver, G. (2008). A representation of t-norms in interval-valued *L*-fuzzy set theory. *Fuzzy Sets and Systems, 159,* 1597-1618.

Deschrijver, G. & Kerre, E.E. (2007). On the position of intuitionistic fuzzy set theory in the framework of theories modelling imprecision. *Information Sciences, 177*, 1860 – 1866.

Hong, D.H. (1998). A note on correlation of interval-valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems, 95*, 113-117.

Hung, W.L. & Wu, J.W. (2002). Correlation of intuitionistic fuzzy sets by centroid method. *Information Sciences, 144*, 219 – 225.

Hwang, C. L. & Yoon, K. (1981). *Multiple Attribute Decision Making: Methods and Applications*. Springer, Berlin, Heideberg, New York, 1981.

Li, D. F. (2005). Multiattribute decision making models and methods using intuitionistic fuzzy sets. *Journal of Computer and System Sciences, 70*, 73-85.

Li, D.F. (2010). Linear programming method for MADM with interval-valued intuitionistic fuzzy sets. *Expert Systems with Applications*, 37, 5939–5945.

Li, D.F. (2010). TOPSIS-based nonlinear-programming methodology for multiattribute decision making with interval-valued intuitionistic fuzzy sets. *IEEE Transactions on Fuzzy Systems*, 18, 299-311

Li, D.F. (2011). Closeness coefficient based nonlinear programming method for interval-valued intuitionistic fuzzy multiattribute decision making with incomplete preference information. *Applied Soft Computing*, 11, 3402–3418.

Li, D.F., Chen, G..H., & Huang, Z.G. (2010). Linear programming method for multiattribute group decision making using IF sets. *Information Sciences*, 180, 1591–1609.

Li, D.F. & Yang, J.B. (2004). Fuzzy linear programming technique for multiattribute group decision making in fuzzy environments. *Information Sciences*, 158, 263–275.

Li, K.W. & Wang, Z. (2010). Notes on "Multicriteria Fuzzy Decision-making Method Based on a Novel Accuracy Function under Interval-valued Intuitionistic Fuzzy Environment", *Journal of Systems Science and Systems Engineering*, 19, 504-508.

Park, J.H., Park, I.Y., Kwun, Y.C., & Tan, X. (2011). Extension of the TOPSIS method for decision making problems under interval-valued intuitionistic fuzzy environment. *Applied Mathematical Modelling*, 35, 2544–2556.

Park, D.G., Kwun, Y.C., Park, J.H., & Park, I.Y. (2009). Correlation coefficient of interval-valued intuitionistic fuzzy sets and its application to multiple attribute group decision making problems. *Mathematical and Computer Modelling*, 50, 1279-1293.

Srinivasan, V. & Shocker, A.D. (1973). Linear programming techniques for multidimensional analysis of preference. *Psychometrica*, 38, 337–342.

Wang, Z., Li, K.W., & Wang, W. (2009). An approach to multiattribute decision making with interval-valued intuitionistic fuzzy assessments and incomplete weights. *Information Sciences, 179*, 3026-3040.

Wang, Z., Li, K.W., & Xu, J. (2011). A mathematical programming approach to multi-attribute decision making with interval-valued intuitionistic fuzzy assessment information. *Expert Systems with Applications*, 38, 12462-12469.

Wang, Z., Wang, L., & Li, K.W. (2011). A linear programming method for interval-valued intuitionistic fuzzy multiattribute group decision making, In *Proceedings of the 2011 Chinese Control and Decision Conference*, 3833-3838, Mianyang, China.

Wei, C. P., Wang, P., Zhang, Y. Z. (2011). Entropy, similarity measure of interval-valued intuitionistic fuzzy sets and their applications. *Information Sciences*, 181, 4273–4286.

Wei, G. (2010). Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making. *Applied Soft Computing* , 10, 423–431.

Wei, G. (2011). Gray relational analysis method for intuitionistic fuzzy multiple attribute decision making. *Expert Systems with Applications*, 38, 11671–11677.

Xu, K., Zhou, J., Gu, R. & Qin, H. (2011). Approach for aggregating interval-valued intuitionistic fuzzy information and its application to reservoir operation. *Expert Systems with Applications*, 38, 9032–9035.

Xu, Z. (2007). Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making. *Control and Decision, 22*, 215-219 (in Chinese).

Xu, Z. (2010). A method based on distance measure for interval-valued intuitionistic fuzzy group decision making. *Information Sciences*, 180, 181-190.

Xu, Z. & Cai, X. (2010). Recent advances in intuitionistic fuzzy information aggregation. *Fuzzy Optimization and Decision Making*, 9, 359-381.

Xu, Z. & Chen, J. (2008). An overview of distance and similarity measures of intuitionistic fuzzy sets. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 16*, 529–555.

Xu, Z. & Yager, R. R. (2008). Dynamic intuitionistic fuzzy multi-attribute making. *International Journal of Approximate Reasoning, 48*, 246-262.

Ye, F. (2010). An extended TOPSIS method with interval-valued intuitionistic fuzzy numbers for virtual enterprise partner selection. *Expert Systems with Applications*, 37, 7050-7055.

Ye, J. (2011). Fuzzy cross entropy of interval-valued intuitionistic fuzzy sets and its optimal decision-making method based on the weights of alternatives. *Expert Systems with Applications*, 38, 6179-6183.

Yue, Z. (2011). An approach to aggregating interval numbers into interval-valued intuitionistic fuzzy information for group decision making. *Expert Systems with Applications*, 38, 6333–6338.

Zadeh, L.A. (1965). Fuzzy sets. *Information and Control*, 8 , 338–356.